Time-optimal reconstruction of Riemannian manifold via boundary electromagnetic measurements

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Abstract

A dynamical Maxwell system is

$$e_t = \operatorname{curl} h, \quad h_t = -\operatorname{curl} e \qquad \text{in } \Omega \times (0, T)$$
$$e|_{t=0} = 0, \quad h|_{t=0} = 0 \qquad \text{in } \Omega$$
$$e_\theta = f \qquad \text{in } \partial\Omega \times [0, T]$$

where Ω is a smooth compact oriented 3-dimensional Riemannian manifold with boundary, $(\,\cdot\,)_\theta$ is a tangent component of a vector at the boundary, $e = e^f(x,t)$ and $h = h^f(x,t)$ are the electric and magnetic components of the solution. With the system one associates a response operator $R^T : f \mapsto -\nu \wedge h^f|_{\partial\Omega \times (0,T)}$, where ν is an outward normal to $\partial\Omega$.

The time-optimal setup of the inverse problem, which is relevant to the finiteness of the wave speed propagation, is: given R^{2T} to recover the part $\Omega^T := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) < T\}$ of the manifold. As was shown by Belishev, Isakov, Pestov, Sharafutdinov (2000), for *small enough* T the operator R^{2T} determines Ω^T uniquely up to isometry.

Here we prove that uniqueness holds for *arbitrary* $T > 0$ and provide a procedure that recovers Ω^T from R^{2T}. Our approach is a version of the boundary control method (Belishev, 1986).

1

0 Introduction

0.1 Maxwell system

Let Ω be a smooth[1] compact oriented three-dimensional Riemannian manifold with the boundary Γ, g the metric tensor on Ω. A dynamical (time-domain) Maxwell system is

$$e_t = \operatorname{curl} h, \quad h_t = -\operatorname{curl} e \qquad \text{in int } \Omega \times (0, T) \qquad (0.1)$$

$$e|_{t=0} = 0, \quad h|_{t=0} = 0 \qquad \text{in } \Omega \qquad (0.2)$$

$$e_\theta = f \qquad \text{on } \Gamma \times [0, T], \qquad (0.3)$$

where $\operatorname{int} \Omega := \Omega \backslash \Gamma$, $(\,\cdot\,)_\theta$ is a tangent component of a vector at Γ, f is a *boundary control*. A solution $\{e, h\}$ describes an electromagnetic wave initiated by the boundary control, $e = e^f(x, t)$ and $h = h^f(x, t)$ being its electric and magnetic components. Since the divergence is an integral of motion, the solution satisfies

$$\operatorname{div} e^f(\,\cdot\,, t) = \operatorname{div} h^f(\,\cdot\,, t) = 0 \qquad \text{in } \Omega \qquad (0.4)$$

for all $t \geq 0$.

An 'input $\to$ output' correspondence of the system is described by a *response operator* $R^T : f \mapsto -\nu \wedge h^f|_{\Gamma \times [0, T]}$, where ν is an outward normal at the boundary, $\wedge$ is the point-wise vector product.

A function (*eikonal*)

$$\tau(x) := \operatorname{dist}(x, \Gamma), \qquad x \in \Omega \qquad (0.5)$$

determines the subdomain (near-boundary layer)

$$\Omega^T := \{x \in \Omega \,|\, \tau(x) < T\}.$$

By the finiteness of the domain of influence principle for the Maxwell equations (shortly: *locality principle*), the relation

$$\operatorname{supp}\{e^f, h^f\} \subset \{(x, t) \,|\, x \in \Omega, \ t \geq \tau(x)\} \qquad (0.6)$$

holds and provides the exact meaning of that the waves propagate with finite speed.

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[1]Everywhere in the paper, 'smooth' means C^∞-smooth.

By the same principle, the *extended problem*

$$e_t = \operatorname{curl} h, \quad h_t = -\operatorname{curl} e \qquad \text{in } D^{2T} \tag{0.7}$$

$$e = 0, \quad h = 0 \qquad \text{in } \{(x,t) \in D^{2T} \mid t < \tau(x)\} \tag{0.8}$$

$$e_\theta = f \qquad \text{on } \Gamma \times [0, 2T] \tag{0.9}$$

in a space-time domain

$$D^{2T} := \{(x,t) \mid x \in \operatorname{int} \Omega^T, \ 0 < t < 2T - \tau(x)\}$$

turns out to be well posed, whereas its solution $\{e^f, h^f\}$ is determined by the part Ω^T of the manifold. With the problem (0.7)–(0.9) one associates an *extended response operator* $R^{2T} : f \mapsto -\nu \wedge h^f|_{\Gamma \times [0, 2T]}$.

0.2   Main result

So, by the locality principle, the operator R^{2T} is determined by the part Ω^T of the manifold Ω. A reasonable question is: **to what extent does the operator R^{2T} determine Ω^T?** As was shown in [2], for small enough T's [2] the operator R^{2T} determines Ω^T uniquely up to isometry. Here this result is strengthened as follows.

Theorem 1 *For any fixed $T > 0$, the operator R^{2T} determines the subdomain Ω^T up to isometry.*

The proof is constructive: given R^{2T} we describe a procedure that provides a manifold $\widetilde{\Omega}^T$ and endows it with a metric tensor $\widetilde{g}$ so that $(\widetilde{\Omega}^T, \widetilde{g})$ turns out to be isometric to (Ω^T, g).

This result was announced in [6]: as was claimed, it can be obtained by straightforward generalization of the approach developed there for the acoustical system to the Maxwell system. However, the proof, which we propose here, is *much simpler*: it is based on quite elementary geometric facts and the version [8] of the fundamental Holmgren-Joihn-Tataru theorem on uniqueness of continuation of solutions to the Maxwell system across a non-characteristic surface. In comparison with the complicated scheme [6], which uses such devices as the Friedrichs extension and Duhamel integral representation, the reconstruction procedure proposed here looks more prospective for numerical realization.

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[2]namely, for $T \leq \operatorname{dist}(c, \Gamma)$, where c is the separation set of Ω w.r.t. Γ

0.3 Comments

- Setting the goal to determine an unknown manifold from its boundary inverse data (here, the response operator R^{2T}), we have to keep in mind the evident nonuiqueness of such a determination: all *isometric* manifolds with the mutual boundary have the same data. Therefore, the only relevant understanding of 'to determine' is to construct a manifold, which possesses the prescribed data [6]. It is what is done in our paper: we provide the manifold $(\widetilde{\Omega}^T, \widetilde{g})$, whose response operator $\widetilde{R}^{2T}$ coincides with the given R^{2T} by construction.

- The literature devoted to inverse problems of electrodynamics is hardly observable and we restrict the list of references by the papers dealing with the *time-optimal* setup of the problem given above. Such an optimality means two things: on the one hand, just by the locality principle, no $R^{2(T-\varepsilon)}$ with $\varepsilon > 0$ determines Ω^T and, on the other hand, to determine Ω^T we need no $R^{2(T+\varepsilon)}$: it suffices to know R^{2T}. The longer is the time interval of observations at the boundary, the bigger is the part of the manifold recovered from the observations. As far as we know, at the moment the boundary control (BC-) method is the only approach, which provides such a locality of reconstruction [3].

- Dealing with the Maxwell system on a manifold, we use and refer to the certain facts and results of [3], which are proved not for this general case but for the Maxwell system in $\Omega \subset \mathbb{R}^3$ with the scalar parameters ε and μ. In all such cases, the generalization is trivial: to get the proof one can just put $\varepsilon = \mu \equiv 1$ and use the intrinsic operations curl, div etc relevant to the Riemannian structure instead of the Euclidean one.

- The authors thank Yu.D.Burago and S.V.Ivanov for very helpful consultations on geometry. We are grateful to I.V.Kubyshkin for kind help in computer graphics.

- The work is supported by the RFBR grants No 08-01-00511 and NSh-4210.2010.1.

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[3]In scalar multidimensional inverse problems, the iterative approach by V.G.Romanov (see [13], [14]) is also time-optimal. Close results for the Maxwell system see in [15].

1 Geometry

1.1 Manifold Ω

We deal with a smooth compact Riemannian manifold Ω: dim $\Omega = 3$, $\Gamma :=$ $\partial\Omega$, d and g are the distance and metric tensor in Ω. Also, the notation (Ω, g) is in use and, if Ω is considered as a metric space (regardless its Riemannian structure), we write (Ω, d).

Convention In what follows, for a subdomain $\Sigma \subset \Omega$, by (Σ, g) we denote this subdomain endowed with the restriction $g|_\Sigma$ of the metric tensor. The notation (Σ, d) means Σ endowed with the *interior* distance that is induced by the tensor g and measured along the curves lying in Σ.

Obviously, (Σ, g) determines (Σ, d) and, as is well known, the converse is also true: given the distance d on Σ one can recover the smooth structure and the metric tensor g, i.e., determine (Σ, g).

The *eikonal* $\tau(\,\cdot\,) : \Omega \to \overline{\mathbb{R}}_+$ is defined by (0.5). For a subset $A \subset \Omega$, we denote its metric neighborhood by

$$\Omega^r[A] := \{x \in \Omega \,|\, d(x, A) < r\}\,, \qquad r > 0\,;$$

in the case of $A = \Gamma$, we write

$$\Omega^r := \Omega^r[\Gamma] = \{x \in \Omega \,|\, \tau(x) < r\}\,.$$

Later, in dynamics, the value

$$T_* := \max_\Omega \tau(\cdot) = \inf\{r > 0 \,|\, \Omega^r = \Omega\}$$

is interpreted as a time needed for waves moving from Γ with the unit speed to fill Ω.

The level sets of the eikonal

$$\Gamma^s := \{x \in \Omega \,|\, \tau(x) = s\}\,, \qquad s \geq 0$$

are the surfaces equidistant to Γ.

For a subset $A \subset \Omega$, by int A we denote the collection of the interior points of the set $A\backslash\Gamma$.
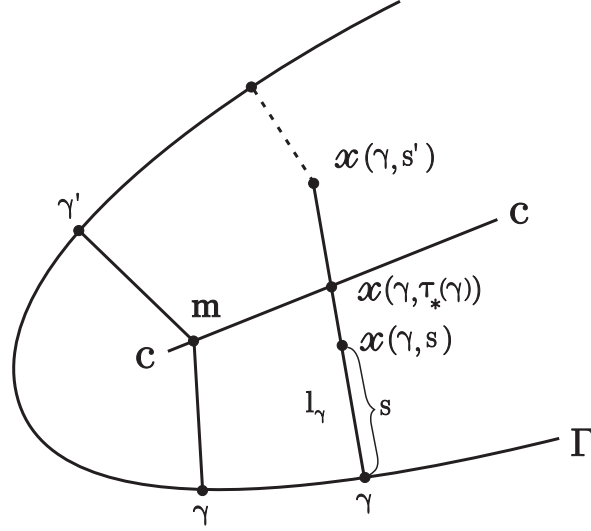
Figure 1: Cut locus

1.2 Cut locus

Recall the definition of a separation set (see, e.g, [10]). Let $l_\gamma[0, s]$ be a segment of the length s of a geodesic l_γ emanating from $\gamma \in \Gamma$ orthogonally to Γ; let $x(\gamma, s)$ be its second endpoint. The value $\tau_*(\gamma)$ is said to be a *critical length* if $\tau(x(\gamma, s)) = s$ for $0 \le s \le \tau_*(\gamma)$ (i.e., $l_\gamma[0, s]$ minimizes the distance between $x(\gamma, s)$ and Γ) and $\tau(x(\gamma, s)) < s$ for $s > \tau_*(\gamma)$ (i.e., $l_\gamma[0, s]$ does not minimize the distance; see the illustration on Fig 1, where $s < \tau_*(\gamma) < s'$). Note that the function $\tau_*(\cdot)$ is continuous on Γ [10]. The point $x(\gamma, \tau_*(\gamma))$ is a *separation point* on l_γ. A set of the separation points

$$c := \bigcup_{\gamma \in \Gamma} x(\gamma, \tau_*(\gamma))$$

is called a separation set (*cut locus*) of Ω w.r.t. Γ.

There is one more way to introduce a cut locus. A point $m \in \Omega$ is said to be *multiple* if it is connected with Γ through more than one shortest geodesics (see Fig 1, where $x(\gamma, s) = x(\gamma', s) = m$ and $s = \tau(m) = \tau_*(\gamma) = \tau_*(\gamma')$). Denote by c_0 the collection of multiple points and define

$$c := \overline{c}_0.$$

Using (mutatis mutandis) the arguments of [12], one can check that this definition is equivalent to the first one .

The cut locus is 'small': the continuity of the function τ_* easily implies $\operatorname{vol} c = 0$.

Note in addition that $\Gamma^s \backslash c$ is a smooth (may be, disconnected) surface in Ω. If $s < d(c, \Gamma)$ then Γ^s is smooth and diffeomorphic to Γ.

1.3   sgc and pattern

For any $x \in \Omega \backslash c$, there is a unique point $\gamma(x) \in \Gamma$ nearest to x and a pair $(\gamma(x), \tau(x))$ is said to be the *semigeodesic coordinates* (sgc) of x. If γ^1, γ^2 are the local coordinates in a neighborhood $\sigma \subset \Gamma$ of $\gamma(x)$, then the functions $\gamma^1(\gamma(\,\cdot\,)), \gamma^2(\gamma(\,\cdot\,)), \tau(\,\cdot\,)$ constitute a coordinate system in a 'tube' $\{x \in \Omega \,|\, \gamma(x) \in \sigma, \ 0 \le \tau(x) < \tau_*(\gamma(x))\}$.

A set

$$\Theta := \{(\gamma(x), \tau(x)) \,|\, x \in \Omega \backslash c\} = \{(\gamma, s) \,|\, \gamma \in \Gamma, \ 0 \le s < \tau_*(\gamma)\} \subset \Gamma \times [0, T_*]$$

is called a *pattern* of Ω, whereas

$$\theta := \bigcup_{\gamma \in \Gamma} (\gamma, \tau_*(\gamma)) \subset \Gamma \times [0, T_*]$$

is its *coast* that is the graph of the function τ_*. The sets

$$\Theta^T := \Theta \cap \{\Gamma \times [0, T)\}, \qquad \theta^T := \theta \cap \{\Gamma \times [0, T)\}$$

are referred to as a pattern of Ω^T and its coast. The patterns are the subgraphs of the continuous functions $\tau_*(\,\cdot\,)$ and $\tau_*^T(\,\cdot\,) := \min\{\tau_*(\cdot), T\}$: see Fig.2 [4].

A map $i : \Omega \backslash c \to \Theta$,

$$i(x) := (\gamma(x), \tau(x))$$

is a diffeomorphism[5]. Its inverse i^{-1} transfers Θ onto $\Omega \backslash c$ by the rule

$$i^{-1}((\gamma, \tau)) = x(\gamma, \tau)$$

and can be extended to $\Theta \cup \theta$ by the same rule. In the sequel, we deal with the extended i^{-1}; it maps $\Theta^T \cup \theta^T$ to Ω^T continuously and surjectively but

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[4]The illustrations on Fig 2 and 3 are taken from [6]

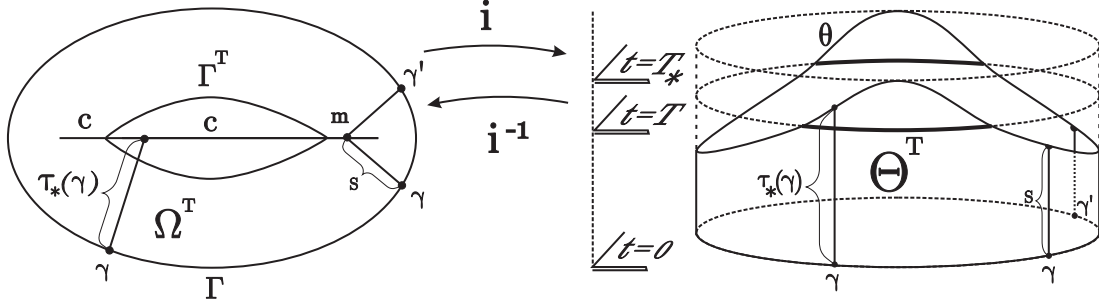[5]Θ is a manifold endowed with the smooth structure of $\Gamma \times [0, T_*]$.

Figure 2: Pattern

not injectively. As is evident, for two points $(\gamma, s), (\gamma', s') \in \Theta^T \cup \theta^T$ the equality

$$i^{-1}\left((\gamma, s)\right) = i^{-1}\left((\gamma', s')\right) \ \ (= m \in \Omega^T \cap c) \tag{1.1}$$

is valid iff they lie at the coast θ^T, $s = s' = \tau(m)$ holds, and their mutual image m is a multiple point (see Fig.2).

1.4 Caps

Fix $\gamma \in \Gamma$, $s > 0$ and a (small) $\varepsilon > 0$; let $\sigma_\varepsilon(\gamma) := \{\gamma' \in \Gamma \mid d(\gamma', \gamma) < \varepsilon\}$ be a portion of the boundary. We say a subdomain

$$\omega_\gamma^{s,\varepsilon} := \Omega^s\left[\sigma_\varepsilon(\gamma)\right] \bigcap \left\{\Omega^s \backslash \Omega^{s-\varepsilon}\right\} =$$
$$= \{x \in \Omega \mid d(x, \sigma_\varepsilon(\gamma)) < s, \ \ s - \varepsilon \leq \tau(x) < s\} \tag{1.2}$$

to be a *cap* and note the monotonicity: $\omega_\gamma^{s,\varepsilon} \subset \omega_\gamma^{s,\varepsilon'}$ as $\varepsilon < \varepsilon'$. Introduce a set

$$\lim_{\varepsilon \to 0} \omega_\gamma^{s,\varepsilon} := \bigcap_{0 < \varepsilon < s} \overline{\omega}_\gamma^{s,\varepsilon}$$

and recall that $x(\gamma, s)$ is defined in sec.1.2; the following result describes the behavior of the caps as $\varepsilon \to 0$.

Proposition 1 *The relation*

$$\lim_{\varepsilon \to 0} \omega_\gamma^{s,\varepsilon} = \begin{cases} x(\gamma, s) & \text{if } s \leq \tau_*(\gamma) \\ \emptyset & \text{if } s > \tau_*(\gamma) \end{cases} \tag{1.3}$$
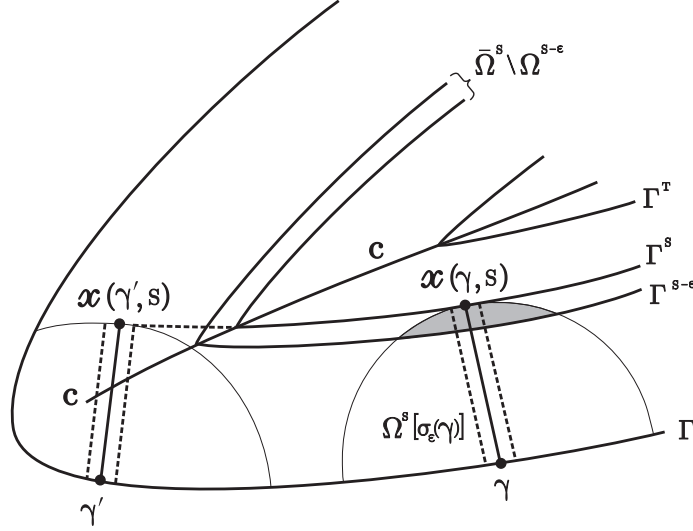
holds.

Figure 3: Cap

Proof see in [6] (Lemma 1).

So, for a given $\gamma \in \Gamma$ and $s > 0$, the cap $\omega_\gamma^{s,\varepsilon}$ either shrinks to the point $x(\gamma, s)$ if $(\gamma, s) \in \Theta \cup \theta$, or terminates (disappears for small enough ε) if $(\gamma, s) \notin \Theta \cup \theta$: see Fig 3, where the cap $\omega_\gamma^{s,\varepsilon}$ is shadowed and $\tau_*(\gamma) > s > \tau_*(\gamma')$) holds. Such a behavior of caps leads to the following evident facts, which we'll use for solving the inverse problem.

Corollary 1 *Let $T > 0$ be fixed. A point $(\gamma, s) \in \Gamma \times [0, T)$ belongs to the set $\Theta^T \cup \theta^T$ iff for any $\varepsilon > 0$ the relation*

$$\omega_\gamma^{s,\varepsilon} \neq \emptyset \tag{1.4}$$

holds; in this case, the inequality $s \leq \tau_*(\gamma)$ is valid. Otherwise, if the family of caps terminates, one has $(\gamma, s) \notin \Theta^T \cup \theta^T$ and, hence, $s > \tau_*(\gamma)$ is valid.

Corollary 2 *Let $\gamma \in \Gamma$ and $(\gamma', s) \in \Theta^T$, so that $x' = x(\gamma', s) \in \Omega^T \backslash c$. For a fixed $r < T$, the inclusion $x' \in \overline{\Omega}^r[\gamma]$ in Ω^T (or, equivalently, the inclusion $(\gamma', s) \in i\left(\overline{\Omega}^r[\gamma]\backslash c\right)$ on Θ^T) holds iff the relation*

$$\omega_{\gamma'}^{s,\varepsilon} \bigcap \Omega^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \neq \emptyset \tag{1.5}$$

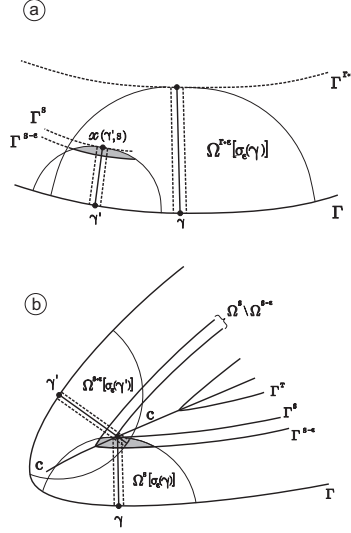*is valid for any $\varepsilon > 0$.*

Figure 4: Relations (1.5) and (1.6)

Corollary 3 *Let the points (γ, s) and (γ', s) belong to the coast θ^T. The equality $i^{-1}\left((\gamma, s)\right) = i^{-1}\left((\gamma', s)\right)$ is valid iff for any $\varepsilon > 0$ the relation*

$$\omega_\gamma^{s,\varepsilon} \bigcap \Omega^{s+\varepsilon}\left[\sigma_\varepsilon(\gamma')\right] \neq \emptyset \tag{1.6}$$

holds (or, equivalently, $\omega_{\gamma'}^{s,\varepsilon} \cap \Omega^{s+\varepsilon}\left[\sigma_\varepsilon(\gamma)\right] \neq \emptyset$).

Corollary 1 is just a convenient reformulation of Proposition 1, whereas Corollaries 2 and 3 easily follow from (1.3) (see the illustrations on Fig.4a,b, where the caps are shadowed).

1.5 Manifold $(\widetilde{\Omega}^T, \widetilde{g})$

Here we prepare a fragment of the future procedure that solves the inverse problem. The fragment is the following construction.

Assume that we are given with the pattern Θ^T endowed with the tensor $g_{\mathrm{sgc}} := (i^{-1})^* g$ [6]. As a metric space, the pair $(\Theta^T, g_{\mathrm{sgc}})$ is isometric to the space $\left(\Omega^T \backslash c, d\right)$, d being understood as the interior distance [7]. Our goal is to make $(\Theta^T, g_{\mathrm{sgc}})$ into an isometric copy of (Ω^T, g).

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[6] in other words, g_{sgc} is the metric tensor g in the semi-geodesic coordinates.

[7] Recall the convention of sec. 1.1!

- **Step 1** Attach the coast θ^T to the pattern Θ^T and extend the metric to $\Theta^T \cup \theta^T$ by continuity. On the extended space, introduce the equivalence

$$\left\{ (\gamma, s) \overset{E}{=} (\gamma', s') \right\} \Leftrightarrow \left\{ i^{-1}(\gamma, s) = i^{-1}(\gamma', s') \right\} ; \qquad (1.7)$$

  let $\widetilde{\Omega}^T := \left[\Theta^T \cup \theta^T \right] / E$ be the quotient set, $\pi : \Theta^T \cup \theta^T \to \widetilde{\Omega}^T$ the projection. Recalling the aforesaid about the map i^{-1}, we see that the equivalence class of a $(\gamma, s) \in \Theta^T \cup \theta^T$ consists of more than one element iff $i^{-1}((\gamma, s))$ is a multiple point. Also, the map $\beta := i^{-1} \circ \pi^{-1} : \widetilde{\Omega}^T \to \Omega^T$ is a well defined bijection. So, as result of sewing the proper points of the coast, we get a set $\widetilde{\Omega}^T$ *bijective* to Ω^T.

- **Step 2** Endow $\widetilde{\Omega}^T$ with the quotient topology[8]. As is easy to recognize, the bijection β is a homeomorphism. Thus, we have a topological space $\widetilde{\Omega}^T$ *homeomorphic* to (Ω^T, g). It remains to endow this space with the relevant Riemannian structure.

- **Step 3** Define a set $\widetilde{c} := \pi(\theta^T) \subset \widetilde{\Omega}^T$, which is the image of the cut locus via bijection β^{-1}. Equip the set $\widetilde{\Omega}^T \backslash \widetilde{c} = \pi(\Theta^T)$ with the metric tensor $\widetilde{g} := (\pi^{-1})^* g_{\text{sgc}}$. As is easy to see, $(\widetilde{\Omega}^T \backslash \widetilde{c}, \widetilde{g})$ is a manifold isometric to $(\Omega^T \backslash c, g)$.

  To extend $\widetilde{g}$ to $\widetilde{c}$ one can take a point $a \in \widetilde{c}$, its neighborhood $\widetilde{\omega} \subset \widetilde{\Omega}^T$ covered by local coordinates u^1, u^2, u^3, find the matrix $\{\widetilde{g}_{jk}(u^1, u^2, u^3)\}_{j,k=1}^3$ in $\widetilde{\omega} \backslash \widetilde{c}$ and then extend the matrix elements to $\widetilde{\omega} \cap \widetilde{c}$ by continuity. We denote the extended tensor by the same symbol $\widetilde{g}$.

As result, we get a Riemannian manifold $(\widetilde{\Omega}^T, \widetilde{g})$, which is *isometric* to (Ω^T, g) by construction.

1.6   Tensor g_{sgc} via distant functions

Return to the starting point of the procedure $(\Theta^T, g_{\text{sgc}}) \Rightarrow (\widetilde{\Omega}^T, \widetilde{g})$ and explain, where the tensor g_{sgc} will be taken from. Let $\Omega^r[\gamma]$ be a semi-ball of the radius $r < T$ with the center $\gamma \in \Gamma$, $i(\Omega^r[\gamma] \backslash c) \subset \Theta^T$ its image in the pattern.

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[8]$A \subset \widetilde{\Omega}^T$ is open iff $\pi^{-1}(A)$ is open in $\Theta^T \cup \theta^T$.

Lemma 1 *The family of the semi-ball images*

$$\mathcal{B}^T := \{i\left(\Omega^r[\gamma]\backslash c\right) \mid \gamma \in \Gamma,\ 0 < r < T\}$$

determines the tensor g_{sgc} *on* Θ^T.

Proof As is evident, to know the family $\mathcal{B}^T$ is to know the distant functions

$$r_a(\gamma, \tau) := d\left(x(\gamma, \tau), a\right)$$

for all $a \in \Gamma$ and $(\gamma, \tau) \in \Theta^T$ provided the distance in the r.h.s. does not exceed T.

Fix a point $(\gamma, \tau) \in \Theta^T$. Take a (small) $\varepsilon > 0$ such that $\eta_\varepsilon[\gamma, \tau] := \sigma_\varepsilon(\gamma) \times (\tau - \varepsilon, \tau + \varepsilon) \subset \Theta^T$ and $r_a|_{\eta_\varepsilon[\gamma,\tau]} < T$ holds for all $a \in \sigma_\varepsilon(\gamma)$. Let γ^1, γ^2 be the local coordinates on $\sigma_\varepsilon(\gamma)$. The metric tensor g_{sgc} on the pattern Θ^T (that is the metric tensor g in sgc) is of the well-known structure

$$g_{\mathrm{sgc}} = \begin{pmatrix} g_{\mathrm{sgc}\,11}(\gamma^1, \gamma^2, \tau) & g_{\mathrm{sgc}\,12}(\gamma^1, \gamma^2, \tau) & 0 \\ g_{\mathrm{sgc}\,21}(\gamma^1, \gamma^2, \tau) & g_{\mathrm{sgc}\,22}(\gamma^1, \gamma^2, \tau) & 0 \\ 0 & 0 & 1 \end{pmatrix} ;$$

by $\{g_{\mathrm{sgc}}^{jk}\}_{j,k=1}^3$ we denote the inverse matrix, *which is of the same structure*.

Choose three *base points* $a_i \in \sigma_\varepsilon(\gamma)$ (see Fig.5). The equalities

$$\left(\frac{\partial r_{a_i}}{\partial \tau}(\gamma, \tau)\right)^2 + g_{\mathrm{sgc}}^{\alpha\beta}(\gamma, \tau) \frac{\partial r_{a_i}}{\partial \gamma^\alpha}(\gamma, \tau) \frac{\partial r_{a_i}}{\partial \gamma^\beta}(\gamma, \tau) = 1 \qquad i = 1,2,3,\ \alpha, \beta = 1, 2,$$

are just the form of writing the well-known fact: the gradient of *any* distant function on Ω is of the norm 1. Consider these equalities as a linear system w.r.t. three unknowns g_{sgc}^{11}, $g_{\mathrm{sgc}}^{12} = g_{\mathrm{sgc}}^{21}$, g_{sgc}^{22}; let $\Delta_{a_1\,a_2\,a_3}(\gamma, \tau)$ be its determinant. As is easy to show, the freedom in the choice of a_1, a_2, a_3 is quite enough to provide $\Delta_{a_1\,a_2\,a_3}(\gamma, \tau) \neq 0$. Hence, varying (if necessary) the position of the base points, one determines from the system the tensor components at the point (γ, τ). $\square$

1.7 Plan

Here we outline the scheme for solving the inverse problem.

Assume that the knowledge of the inverse data (operator R^{2T}) enables one to check the relations (1.4), (1.5), and (1.6). If so, we can realize the following construction:
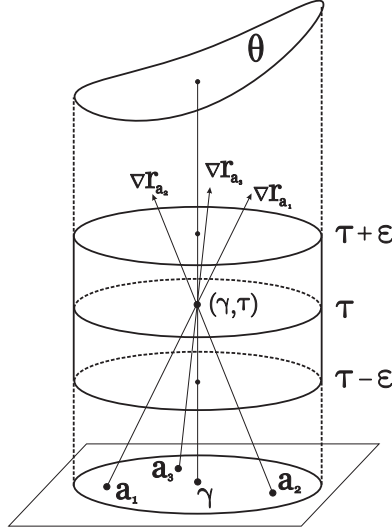
Figure 5: Recovering g_{sgc}

- Select the points $(\gamma, s) \in \Gamma \times [0, T)$, for which (1.4) holds and thus recover the function $\tau_*^T(\gamma)$, $\gamma \in \Gamma$, which determines the pattern $\Theta^T \subset \Gamma \times [0, T)$ and its coast θ^T

- Fix $\gamma \in \Gamma$, take $(\gamma', s) \in \Theta^T$, $r > 0$ available for Corollary 2, and check by (1.5) whether $(\gamma', s) \in i\left(\overline{\Omega}^r[\gamma] \backslash c\right)$ holds. Varying (γ', s), recover $i\left(\overline{\Omega}^r[\gamma] \backslash c\right)$. Then, varying γ and r, determine the family $\mathcal{B}^T$ of semi-ball images on Θ^T. By Lemma 1, this family determines the metric tensor g_{sgc} on Θ^T; thus, we get the manifold $(\Theta^T, g_{\mathrm{sgc}})$.

- Using (1.6) as a sewing criterion, introduce the equivalence E. Given $(\Theta^T, g_{\mathrm{sgc}})$ and E, construct the isometric copy $(\widetilde{\Omega}^T, \widetilde{g})$ of the manifold (Ω^T, g) by the procedure of sec 1.5.

At this point, one can claim that the inverse problem is solved: an isometric copy of the original manifold is determined from the inverse data. It remains to explain, how to check (1.4)–(1.6) via R^{2T}.

2 Dynamics

2.1 Vector analysis

Begin with recalling the definitions of operations on vector fields in a 3d-manifold (see [16] for detail). As above, g is the metric tensor (2-form on vector fields). We assume that Ω is oriented and denote by μ the volume 3-form.

A *scalar product* ' $\cdot$ ' : {fields} $\times$ {fields} $\rightarrow$ {functions} is defined by $a \cdot b = g(a, b)$.

A *vector product* $\wedge$: {fields} $\times$ {fields} $\rightarrow$ {fields} is defined by $g(a \wedge b, c) = \mu(a, b, c)$.

For a field a, its *conjugate 1-form* $a_\sharp$ is defined by $a_\sharp(b) = g(a, b)$. Conversely, for an 1-form ϕ one defines its *conjugate field* $\phi^\sharp$ by $\phi(b) = g(b, \phi^\sharp)$.

A *gradient* ∇ : {functions} $\rightarrow$ {fields} is $\nabla u = (\mathrm{d}u)^\sharp$, where d is the exterior derivative.

A *divergence* div : {fields} $\rightarrow$ {functions} acts by $\operatorname{div} a = \star \mathrm{d} \star a_\sharp$, where $\star$ is the Hodge operator.

A *curl* is defined as a map curl : {fields} $\rightarrow$ {fields}, $\quad \operatorname{curl} a = (\star \mathrm{d} a_\sharp)^\sharp$.

The basic identities are $\operatorname{div}\operatorname{curl} = 0$ and $\operatorname{curl}\nabla = 0$.

Let ν be the outward unit normal on Γ, μ_Γ the (induced) surface form on Γ; recall the Green formula

$$\int\limits_\Omega u \operatorname{div} a \; \mu = \int\limits_\Gamma u \, a \cdot \nu \; \mu_\Gamma - \int\limits_\Omega a \cdot \nabla u \; \mu \,. \tag{2.1}$$

2.2 System α^T

In what follows, we deal not with the complete Maxwell system (0.1)–(0.3) but its electric subsystem that is obtained by eliminating the magnetic component [9]; we write it in the form

$$
\begin{aligned}
e_{tt} &= -\operatorname{curl}\operatorname{curl} e & &\text{in int } \Omega^T \times (0, T) & &(2.2) \\
e &= 0 & &\text{in } \{(x, t) \in \Omega^T \times (0, T) \,|\, t < \tau(x)\} & &(2.3) \\
e_\theta &= f & &\text{on } \Gamma \times [0, T] \,, & &(2.4)
\end{aligned}
$$

---

[9] A reason to single out the subsystem is that the components e^f and h^f are not quite independent: for times $t < T_*$ the magnetic component is determined by electric one [3].

and refer to as the *dynamical system* α^T. Such a form corresponds to the locality principle: the solution e^f is determined by the part (Ω^T, g) only.

Let us equip α^T with standard control theory attributes: spaces and operators.

Outer space A control f in (2.4) is a time-dependent tangent vector field on Γ. Let $\mathcal{T}$ be a space of square summable tangent fields on Γ; the space of controls $\mathcal{F}^T := L_2\{[0,T]; \mathcal{T}\}$ with the product

$$(f, g)_{\mathcal{F}^T} = \int\limits_0^T dt \int\limits_\Gamma f(\gamma, t) \cdot g(\gamma, t) \, \mu_\Gamma$$

is called *outer*. It contains an extending family of subspaces

$$\mathcal{F}^{T,\xi} := \left\{ f \in \mathcal{F}^T \mid f|_{0 \leq t < T-\xi} = 0 \right\}, \qquad 0 \leq \xi \leq T$$

formed by delayed controls; the parameter ξ is regarded as the action time, whereas $T - \xi$ is the delay.

Also, the class of controls

$$\mathcal{F}_+^T := L_2\left([0,T]; \mathcal{T} \cap \vec{H}^{\frac{1}{2}}(\Gamma)\right),$$

where $\vec{H}^{\frac{1}{2}}(\Gamma)$ is the Sobolev vector space[10], and the smooth class

$$\mathcal{M}^T := \{ f \in C^\infty\left([0,T]; \mathcal{T} \cap C^\infty(\Gamma)\right) \mid \operatorname{supp} f \subset (0, T]\}$$

are in use. Note that by the definition of the second class, each $f \in \mathcal{M}^T$ vanishes near $t = 0$. The class $\mathcal{F}_+^T$ is a normed space w.r.t. the relevant norm, whereas $\mathcal{M}^T$ is dense in $\mathcal{F}^T$ and in $\mathcal{F}_+^T$.

For $f \in \mathcal{M}^T$, the problem (2.2)–(2.4) has a unique classical (smooth) solution e^f, which is a time-dependent vector field in Ω. Being initially defined on $\mathcal{M}^T$, the map $f \mapsto e^f$ acts continuously from $\mathcal{F}_+^T$ to the space $C\left([0,T], \vec{L}_2(\Omega^T)\right)$ (see [9]). Hence, it can be extended to the class $\mathcal{F}_+^T$. In what follows we assume that such an extension is done and regard its images as *generalized solutions* of the problem (2.2)–(2.4).

---

[10] as is customary on manifolds, we assume $\vec{H}^{\frac{1}{2}}(\Gamma)$ to be endowed with one of the equivalent relevant Sobolev norms

Inner space In control theory, the solution e^f is referred to as a *trajectory* of the system α^T, whereas $e^f(\,\cdot\,,t)$ is a *state* at the moment t. By (0.4), the states are divergence-free fields, whereas by (0.6) one has

$$\operatorname{supp} e^f(\,\cdot\,,t) \subset \overline{\Omega^t} \subset \overline{\Omega}^T\,, \tag{2.5}$$

i.e, the trajectory does not leave $\overline{\Omega}^T$. Thus, the natural candidate for the role of the space of states is

$$\mathscr{J}^T := \operatorname{clos}\{y \in \vec{L}_2(\Omega^T)\,|\, \operatorname{supp} y \subset \Omega^T,\ \operatorname{div} y = 0\ \text{in}\ \Omega^T\}\,,$$

where clos is the closure in the vector fields space $\vec{L}_2(\Omega^T)$ with the product

$$(y,v)_{\vec{L}_2(\Omega^T)} = \int\limits_{\Omega^T} y \cdot v\ \mu$$

and $\operatorname{div} y = 0$ is understood in the sense of distributions. The space $\mathscr{J}^T$ is called *inner*, the waves $e^f(\,\cdot\,,t)$ are its elements.

The inner space contains an extending family of subspaces

$$\mathscr{J}^\xi := \operatorname{clos}\{y \in \mathscr{J}^T\,|\,\operatorname{supp} y \subset \Omega^\xi\}\,, \qquad 0 \le \xi \le T\,.$$

By the definitions and (2.5), one has

$$e^f(\,\cdot\,,t) \in \mathscr{J}^t\,, \qquad 0 \le t \le T\,. \tag{2.6}$$

Control operator An 'input $\to$ state' correspondence in the system α^T is realized by a *control operator* $W^T : \mathscr{F}^T \to \mathscr{J}^T,\ \operatorname{Dom} W^T = \mathscr{F}^T_+,$

$$W^T f := e^f(\,\cdot\,,T)\,,$$

which is a closable unbounded operator (see [9]). Hence, it can be extended up to a closed operator, what we assume to be done[11] and denote the closure by the same symbol W^T.

As each closed operator, the control operator can be represented in the form of the polar decomposition

$$W^T = \Phi^T\,|W^T|\,, \tag{2.7}$$

---

[11]However, the precise description of $\operatorname{Dom} \overline{W}^T$ is not known yet.

where $|W^T| := \left[\left(W^T\right)^* W^T\right]^{\frac{1}{2}}$ is the modulus of W^T and Φ^T is an isometry from $\operatorname{Ran}|W^T|$ onto $\operatorname{Ran} W^T$ (see, e.g., [7]).

Response operator An 'input $\to$ output' correspondence is described by a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \mathcal{M}^T$,

$$\left(R^T f\right)(\gamma, t) := \nu(\gamma) \wedge \operatorname{curl} \int_0^t e^f(\gamma, s) \, ds \,, \qquad (\gamma, t) \in \Gamma \times [0, T] \,,$$

where ν is an outward normal, the expression in the right hand side being equal to $-\nu \wedge h^f|_{\Gamma \times [0,T]}$. This operator is unbounded but closable. The latter is a simple consequence of the following fact: it can be shown that R^T acts continuously from $\mathcal{F}^T$ to a Sobolev negative space $\vec{H}^{-p}(\Gamma \times [0, T])$ with a big enough $p > 0$.

In the mean time, the locality principle enables one to extend the problem (2.2)–(2.4) to the system

$$e_{tt} = -\operatorname{curl} \operatorname{curl} e \qquad\qquad \text{in } D^{2T} \qquad\qquad (2.8)$$

$$e = 0 \qquad\qquad \text{in } \{(x, t) \in D^{2T} \,|\, t < \tau(x)\} \qquad (2.9)$$

$$e_\theta = f \qquad\qquad \text{on } \Gamma \times [0, 2T] \qquad\qquad (2.10)$$

(see (0.7)–(0.9)) and introduce an *extended response operator* $R^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $\operatorname{Dom} R^{2T} = \mathcal{M}^{2T}$,

$$\left(R^{2T} f\right)(\gamma, t) := \nu(\gamma) \wedge \operatorname{curl} \int_0^t e^f(\gamma, s) \, ds \,, \qquad (\gamma, t) \in \Gamma \times [0, 2T] \,.$$

This operator is also determined by the part (Ω^T, g) of the manifold and, hence, can be regarded as an intrinsic object of the system α^T [12].

Connecting form A bilinear form $c^T : \mathcal{F}^T \times \mathcal{F}^T \to \mathbb{R}$, $\operatorname{Dom} c^T = \mathcal{F}^T_+ \times \mathcal{F}^T_+$,

$$c^T[f, g] := \left(e^f(\,\cdot\,, T), e^g(\,\cdot\,, T)\right)_{\mathscr{J}^T} = \left(W^T f, W^T g\right)_{\mathscr{J}^T}$$

is called *connecting*. By closability of W^T, the form c^T is also closable [7].

The following fact is one of the key points of the BC-method: it is used in all of its versions [1], [6].

---

[12]More about the extended (continued) response operator see in [5].

Proposition 2 *The connecting form c^T is determined by the response operator R^{2T}.*

Moreover, c^T can be expressed through R^{2T} explicitly as follows. Let $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ be the operator that extends the controls, as functions of t, from $[0, T]$ to $[0, 2T]$ by oddness w.r.t. $t = T$. Introduce the class $\mathcal{M}^{T,0} := \{ f \in \mathcal{M}^T \mid S^T f \in \mathcal{M}^{2T} \}$; it is dense in $\mathcal{F}_+^T$ and $S^T \mathcal{M}^{T,0} \subset \operatorname{Dom} R^{2T}$ holds. The relation

$$c^T[f, g] = \left(-\frac{1}{2} \left(S^T \right)^* R^{2T} S^T f,\ g \right)_{\mathcal{F}^T} \tag{2.11}$$

is valid for $f \in \mathcal{M}^{T,0}, g \in \mathcal{F}_+^T$ (see [3]).

Corollary 4 *The response operator R^{2T} determines the operator $|W^T|$.*

Indeed, R^{2T} determines c^T, and one has

$$c^T[f, g] = \left(W^T f,\ W^T g \right)_{\mathcal{J}^T} = \langle \text{see } (2.7) \rangle = \left(|W^T| f,\ |W^T| g \right)_{\mathcal{F}^T}, \tag{2.12}$$

whereas the *positive* operator $|W^T|$ is determined by the latter form in the r.h.s..

2.3 Reachable sets

Fix an open $\sigma \subset \Gamma$; a linear set

$$\mathcal{U}_{\text{reach}}^\xi[\sigma] := \left\{ W^T f \mid f \in \mathcal{M}^T,\ \operatorname{supp} f \subset \sigma \times (T - \xi, T] \right\}$$

is called *reachable* (from σ, at the time ξ). So, the reachable set consists of electric waves $e^f(\,\cdot\,, T)$ produced by all smooth delayed controls, which act from the part of boundary σ, the action time being equal to ξ. Its closure (in $\mathcal{J}^T$)

$$\mathcal{U}^\xi[\sigma] := \operatorname{clos} \mathcal{U}_{\text{reach}}^\xi[\sigma]$$

is said to be a *reachable subspace*. Also, we denote shortly $\mathcal{U}^\xi[\Gamma] =: \mathcal{U}^\xi$.

For a measurable subset $A \subset \Omega^T$, we define a subspace

$$\mathcal{J}\langle A \rangle := \operatorname{clos} \left\{ y \in \mathcal{J}^T \mid \operatorname{supp} y \subset A \right\}.$$

By the finiteness of the wave speed propagation, each wave belonging to $\mathcal{U}_{\text{reach}}^\xi[\sigma]$ is supported in the metric neighborhood $\Omega^\xi[\sigma]$. Hence, the embedding

$$\mathcal{U}^\xi[\sigma] \subset \mathcal{J}\langle \Omega^\xi[\sigma] \rangle \tag{2.13}$$

holds. A structure of the reachable sets and subspaces, as well as the character of the embedding (2.13) is the subject of the boundary control theory. For our goals, the following fact is of crucial value.

Proposition 3 *Any field $h \in \mathcal{J}\langle \Omega^\xi[\sigma] \rangle \ominus \mathcal{U}^\xi[\sigma]$ is smooth and satisfies*

$$\operatorname{curl} h \,=\, 0 \qquad\qquad (2.14)$$

in $\operatorname{int} \Omega^\xi[\sigma]$.

Proof see in [3], [6] and the remark in the third item of sec 0.3; more about properties of the reachable sets see in [4]. Note that the derivation of (2.14) relays upon the fundamental Holmgren-John-Tataru theorem on uniqueness of continuation of the solutions to the Maxwell system across a non-characteristic surface [8].

Corollary 5 *If a subdomain $\omega \subset \operatorname{int} \Omega^\xi[\sigma]$ is homeomorphic to an open ball in $\mathbb{R}^3$, then the relation*

$$\mathcal{J}\langle \omega \rangle \,\subset\, \mathcal{U}^\xi[\sigma] \qquad\qquad (2.15)$$

holds.

Indeed, by (2.14) any $h \in \mathcal{J}\langle \Omega^\xi[\sigma] \rangle \ominus \mathcal{U}^\xi[\sigma]$ can be represented locally as $h|_\omega = \nabla p$. Hence, for $y \in \mathcal{J}\langle \omega \rangle \subset \mathcal{J}\langle \Omega^\xi[\sigma] \rangle$ one has

$$(h,y)_{\mathcal{J}^T} = \int_\omega \nabla p \cdot y \; \mu = \langle \text{see } (2.1) \rangle \,=\, 0\,,$$

i.e., $y \in \mathcal{U}^\xi[\sigma]$ and we arrive at (2.15). This result shows that the reachable subspace $\mathcal{U}^\xi[\sigma]$ is always rich enough. Moreover, in a relevant sense, the generic case is that the defect subspace $\mathcal{J}\langle \Omega^\xi[\sigma] \rangle \ominus \mathcal{U}^\xi[\sigma]$ is finite dimensional, its dimension being determined by topology of $\Omega^\xi[\sigma]$ (see, e.g., [4]).

3   Inverse problem

3.1   Wave caps

Here the main technical tool for solving the inverse problem is introduced.

In parallel with the definition (1.2), introduce a subspace

$$w_\gamma^{s,\varepsilon} := \mathcal{U}^s[\sigma_\varepsilon(\gamma)] \bigcap \{\mathcal{U}^s \ominus \mathcal{U}^{s-\varepsilon}\} \qquad\qquad (3.1)$$

and call it a *wave cap*. The correspondence between the wave and 'space' caps is as follows. Assume that γ, ε, and s are chosen so that the cap $\omega_\gamma^{s,\varepsilon}$ is nonempty.

- *Each $y \in w_\gamma^{s,\varepsilon}$ is supported in $\overline{\omega}_\gamma^{s,\varepsilon}$.* Indeed, by (2.14) with $\sigma = \Gamma$, the orthogonality $y \perp \mathcal{U}^{s-\varepsilon}$ implies that the field y is harmonic[13] in $\Omega^{s-\varepsilon}$. In the mean time, y is supported in the neighborhood $\overline{\Omega}^s[\sigma_\varepsilon(\gamma)]$ and, whence, vanishes out of this neighborhood. By the uniqueness of the harmonic continuation, y has to vanish everywhere in $\Omega^{s-\varepsilon}$ and, thus, to be supported in the space cap.

- Fix an x and a (small) ball $\Omega^\delta[x] \subset \operatorname{int} \omega_\gamma^{s,\varepsilon}$. As is easy to see from (2.15), the subspace $\mathcal{J}\langle \Omega^\delta[x] \rangle$ is embedded in the r.h.s. of (3.1). Hence, *the embedding $\mathcal{J}\langle \Omega^\delta[x] \rangle \subset w_\gamma^{s,\varepsilon}$ holds* and shows that the wave cap is a rich enough nonzero subspace.

As one can easily conclude from the above-mentioned facts, the space and wave caps behave in one and the same way: the equivalence

$$\left\{ \omega_\gamma^{s,\varepsilon} \neq \emptyset \right\} \Longleftrightarrow \left\{ w_\gamma^{s,\varepsilon} \neq \{0\} \right\} \tag{3.2}$$

holds.

Return to Corollaries 1–3. The 'subdomain $\leftrightarrow$ subspace' arguments quite analogous to the ones that have led to (3.2) imply the equivalences

$$\left\{ \omega_{\gamma'}^{s,\varepsilon} \bigcap \Omega^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \neq \emptyset \right\} \Longleftrightarrow \left\{ w_{\gamma'}^{s,\varepsilon} \bigcap \mathcal{U}^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \neq \{0\} \right\} \tag{3.3}$$

$$\left\{ \omega_\gamma^{s,\varepsilon} \bigcap \Omega^{s+\varepsilon}[\sigma_\varepsilon(\gamma')] \neq \emptyset \right\} \Longleftrightarrow \left\{ w_\gamma^{s,\varepsilon} \bigcap \mathcal{U}^{s+\varepsilon}[\sigma_\varepsilon(\gamma')] \neq \{0\} \right\}. \tag{3.4}$$

3.2   Model caps

Return to the polar decomposition (2.7).

Fix an open $\sigma \subset \Gamma$. In the outer space $\mathcal{F}^T$, introduce a linear set

$$|\mathcal{U}|_{\mathrm{reach}}^\xi[\sigma] := \left\{ |W|^T f \mid f \in \mathcal{M}^T, \ \operatorname{supp} f \subset \sigma \times (T - \xi, T] \right\}$$
$$= (\Phi^T)^* \mathcal{U}_{\mathrm{reach}}^\xi[\sigma] \tag{3.5}$$

---

[13] i.e., $\operatorname{curl} y = 0$ and $\operatorname{div} y = 0$; see, e.g., [16]

and a subspace

$$|\mathcal{U}|^\xi[\sigma] := \text{clos } |\mathcal{U}|^\xi_{\text{reach}}[\sigma] = (\Phi^T)^* \mathcal{U}^\xi[\sigma] \qquad (3.6)$$

(the closure in $\mathcal{F}^T$) that we call the *model* reachable set and subspace respectively. Subsequently, define a *model wave cap*

$$|w|^{s,\varepsilon}_\gamma := |\mathcal{U}|^s[\sigma_\varepsilon(\gamma)] \bigcap \{|\mathcal{U}|^s \ominus |\mathcal{U}|^{s-\varepsilon}\} = (\Phi^T)^* w^{s,\varepsilon}_\gamma. \qquad (3.7)$$

Since the map $(\Phi^T)^*$ is an isometry, the subspaces in the right hand sides of the equivalences (3.3) and (3.4) can be replaced by their model copies, i.e., the images through this map. Indeed, for instance, by virtue of the evident equality

$$|w|^{s,\varepsilon}_{\gamma'} \bigcap |\mathcal{U}|^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] = (\Phi^T)^* \left(w^{s,\varepsilon}_{\gamma'} \bigcap \mathcal{U}^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \right)$$

the conditions

$$w^{s,\varepsilon}_{\gamma'} \bigcap \mathcal{U}^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \neq \{0\} \qquad \text{and} \qquad |w|^{s,\varepsilon}_{\gamma'} \bigcap |\mathcal{U}|^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \neq \{0\}$$

are equivalent.

Return once again to Corollaries 1–3. The aforesaid enables one to reformulate them as follows.

Corollary 6 *Let $T > 0$ be fixed. A point $(\gamma, s) \in \Gamma \times [0, T)$ belongs to the set $\Theta^T \cup \theta^T$ iff for any $\varepsilon > 0$ the relation*

$$|w|^{s,\varepsilon}_\gamma \neq \{0\} \qquad (3.8)$$

holds; in this case, the inequality $s \leq \tau_*(\gamma)$ is valid. Otherwise, if the family of caps terminates, one has $(\gamma, s) \notin \Theta^T \cup \theta^T$ and, hence, $s > \tau_*(\gamma)$ is valid.

Corollary 7 *Let $\gamma \in \Gamma$ and $(\gamma', s) \in \Theta^T$, so that $x' = x(\gamma', s) \in \Omega^T \backslash c$. For a fixed $r < T$, the inclusion $x' \in \overline{\Omega}^r[\gamma]$ in Ω^T (or, equivalently, the inclusion $(\gamma', s) \in i\left(\overline{\Omega}^r[\gamma] \backslash c\right)$ on Θ^T) holds iff the relation*

$$|w|^{s,\varepsilon}_{\gamma'} \bigcap |\mathcal{U}|^{r+\varepsilon}[\sigma_\varepsilon(\gamma)] \neq \{0\} \qquad (3.9)$$

holds for any $\varepsilon > 0$.

Corollary 8 *Let the points (γ, s) and (γ', s) belong to the coast θ^T. The relation $(\gamma, s) \overset{E}{=} (\gamma', s)$ is valid iff for any $\varepsilon > 0$ the relation*

$$|w|_\gamma^{s,\varepsilon} \bigcap |\mathcal{U}|^{s+\varepsilon} [\sigma_\varepsilon(\gamma')] \neq \{0\} \qquad (3.10)$$

holds (or, equivalently, $|w|_{\gamma'}^{s,\varepsilon} \cap |\mathcal{U}|^{s+\varepsilon} [\sigma_\varepsilon(\gamma)] \neq \{0\}$).

The key fact is that the subspaces in the l.h.s. of (3.8)–(3.10) are determined by the operator $|W^T|$ and, whence, by the inverse data (operator R^{2T}).

3.3 Reconstruction

Now, to solve the inverse problem it suffices just to summarize our previous considerations, which we present in the form of the following procedure. Recall the starting point: *we are given with the extended response operator R^{2T}.*

Step 1 Recover the connecting form c^T from (2.11) and determine the operator $|W^T|$ (see Corollary 4, (2.12)).

Step 2 Determine the sets $|\mathcal{U}|_{\text{reach}}^\xi[\sigma]$, subspaces $|\mathcal{U}|^\xi[\sigma]$, and the model caps $|w|_\gamma^{s,\varepsilon}$ by the definitions (3.5), (3.6), and (3.7) respectively for those $\sigma, \varepsilon, \gamma, \xi$, which are used in Corollaries 6–8.

Step 3 Checking the relation (3.8), recover the pattern Θ^T and its coast θ^T.

Step 4 By checking (3.9), determine the family of ball images $\mathcal{B}^T$ and recover the metric tensor g_{sgc} on the pattern (see Lemma 1).

Step 5 By the use of (3.10), recover the equivalence E. The collection Θ^T, θ^T, E determines the manifold $(\widetilde{\Omega}^T, \widetilde{g})$, which can be constructed by means of the procedure described in sec 1.5.

As a result, the operator R^{2T} determines the manifold $(\widetilde{\Omega}^T, \widetilde{g})$, which is isometric to the manifold (Ω^T, g) by construction. In other words, it determines (Ω^T, g) up to isometry that proves Theorem 1. Moreover, identifying the points by

$$\Omega^T \supset \Gamma \ni \gamma \equiv \pi\left((\gamma, 0)\right) \in \partial\widetilde{\Omega}^T \subset \widetilde{\Omega}^T$$

we can easily check that the response operator $\widetilde{R}^{2T}$ of the constructed manifold is identical to R^{2T}. The latter motivates to refer to $(\widetilde{\Omega}^T, \widetilde{g})$ as a *canonical representative* of the class of isometric manifolds possessing the given inverse data R^{2T}.

In conclusion, note the following. It is not inconvincible that the above described procedure $R^{2T} \Rightarrow (\widetilde{\Omega}^T, \widetilde{g})$ is available for numerical realization. The principal problem (and difficulty) is to get good enough simulation of the wave caps $w_\gamma^{s,\varepsilon}$ that is to provide a rich enough set of waves $e^f(\,\cdot\,, T)$ concentrated near the point $x(\gamma, s)$. The sharper is the concentration, the better is the reconstruction of the pattern, distant functions r_a, etc. Such a principle is traced in all versions of the BC-method.

References

[1] M.I. Belishev. Boundary control in reconstruction of manifolds and metrics (the BC method). *Inverse Problems*, 13(5): R1–R45, 1997.

[2] M.I.Belishev, V.M.Isakov, L.N.Pestov, and V.A.Sharafutdinov. On the Reconstruction of Metrics via External Electromagnetic Measurements. *Doklady Mathematics*, 61 (2000), No 3, 353–356.

[3] M.I.Belishev, A.K.Glasman. Dynamical inverse problem for the Maxwell system: recovering the velocity in a regular zone (the BC–method). *St.-Petersburg Math. Journal*, 12 (2001), No 2: 279–316.

[4] M.I.Belishev and A.K.Glasman. Boundary control of the Maxwell dynamical system: lack of controllability by topological reasons. *ESAIM COCV*, 5 (2000), 207–217.

[5] M.I.Belishev. Dynamical systems with boundary control: models and characterization of inverse data. *Inverse Problems*, 17 (2001), 659–682.

[6] M.I.Belishev. Recent progress in the boundary control method. *Inverse Problems*, 23 (2007), no 5, R1–R67.

[7] M.S. Birman, M.Z. Solomyak. Spectral Theory of Self-Adjoint Operators in Hilbert Space. *D.Reidel Publishing Comp.*, 1987.

[8] M.Eller, V.Isakov, G.Nakamura, D.Tataru. Uniqueness and stability in the Cauchy problem for Maxwell's and elasticity systems. *Nonlinear PDE and Applications, Eds. D.Cioranescu, J-L. Lions, College de France Seminar*, 14, 329–349, "Studies in Mathematics and its applications", v.31, North-Holland, Elsevier Science, 2002.

[9] M.Eller. Symmetric hyperbolic systems with boundary conditions that do not satisfy the Kreiss-Sakamoto condition. *Applicationes Mathematicae*, 35 (2008), 323-333.

[10] D.Gromol, W.Klingenberg, W.Meyer. Riemannische Geometrie im Grossen, *Berlin, Springer*, 1968.

[11] V.Isakov. Inverse problems for partial differential equations. *Appl. Math. Studies, Springer*, v. 127, 2006.

[12] W.Klingenberg. Riemannian geometry. *Series YB de Gruyer: Studies in Mathematics*, vol 1, 1982.

[13] V.G.Romanov. On solvability of the inverse problems for the hyperbolic equations in a class of functions analytical with respect to a part of variables. *Doklady Akad. Nauk SSSR*, 304 (4): 51–67, 1989 (in Russian).

[14] V.G.Romanov. Stability in inverse problems. *Moskva, Nauchnyi Mir*, 2005 (in Russian).

[15] V.G.Romanov, T.N.Pukhnacheva. Solution stability theorem in coefficients determination problem for the Maxwell equations system. *Ill-Posed Problems; Ed. M.M.Lavrent'ev, Novosibirsk, SO AN SSSR*, 1988, 64–86 (in Russian).

[16] G.Schwarz. Hodge decomposition - a method for solving boundary value problems. *Lecture notes in Math.*, 1607. Springer–Verlag, Berlin, 1995.